

# Critical Exponents and Universality in Bose-Einstein Condensation

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## ABSTRACT

We report calculation of critical exponents corresponding to three thermodynamic functions, *viz.* chemical potential, condensate fraction and specific heat of a non-interacting Bose gas confined in an isotropic harmonic oscillator trap in one-, two- and three-dimensions. Critical exponents are also calculated for bosons interacting through van der Waals potential and confined in a three-dimensional harmonic trap. These exponents are found to be independent of the size of the system and whether the bosons are interacting or not, demonstrating their universality property. We also find that a continuous phase transition occurs even in one-dimension, as shown by a rapid change of the thermodynamic functions at the critical point. This is in sharp contrast with the result found in the text book treatment of replacing the sum over occupations of discrete states by an integral. The two- and three-dimensional results agree with the well known results in the *thermodynamic* limit, *i.e.* when the number of bosons in the trap approaches infinity. In the same limit, the criticality disappears for the one-dimensional trap, which again agrees with the standard result.

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Bose-Einstein condensation (BEC) is the transition process, in which a macroscopic fraction of bosons goes into the lowest energy state, below a certain critical temperature ( $T_c$ ) [1]. It was predicted by Einstein in 1925, based on Bose's explanation

of black body radiation. The BEC is generally considered as a phase transition in the thermodynamic limit. In standard text books [1, 2], it is shown that BEC is possible in an ideal uniform (*i.e.* not confined) Bose gas at a finite temperature in three and higher dimensions, while in two-dimension the transition temperature becomes zero and no BEC transition is possible in one-dimension. In a uniform ideal (non-interacting) Bose gas in three-dimension, the condensate fraction (defined as  $\frac{N_0}{N}$ , where  $N_0$  is the number of bosons in the lowest energy state and  $N$  is the total number of bosons) at a temperature  $T < T_c$  is given by [1]

$$\frac{N_0}{N} = 1 - \left(\frac{T}{T_c}\right)^{\frac{3}{2}}, \quad (1)$$

while the specific heat is given by [1, 3]

$$C_N(T) = \frac{15}{4} N k_B \frac{\zeta(\frac{5}{2})}{\zeta(\frac{3}{2})} \left(\frac{T}{T_c}\right)^{\frac{3}{2}}, \quad (2)$$

where  $k_B$  is the Boltzmann constant and  $\zeta(x) = \sum_{n=1}^{\infty} n^{-x}$  is the Riemann zeta function. Both condensate fraction and  $C_N$  are continuous, while the first derivatives are discontinuous at  $T = T_c$  for the ideal Bose gas [1]. This means that in the thermodynamic limit ( $N \rightarrow \infty$ ) BEC is a continuous phase transition process. The situation is quite different if the Bose gas is non-uniform, *i.e.*, confined in a suitable trap [4]. Commonly used experimental traps are spherical or axially symmetric harmonic oscillator traps. In such cases,  $C_N(T)$  has a *finite* discontinuity at  $T = T_c$  [1, 4]. Thus, the corresponding phase transition is of first order. However, it is important to note that in the usual text book treatment, the sum representing occupation of single particle levels is replaced by an integral, involving the density of states which is obtained by a semi-classical approximation. In doing so, the occupation of the lowest energy level is taken separately. This approximation is valid only in the thermodynamic limit of an ideal uniform Bose gas (having continuous energy levels). In a potential trap, with discrete energy levels, this replacement is not justified for a finite number of atoms. Replacing the sum by an integration including semi-classical density of states, Bagnato and Kleppner [5] showed that BEC is possible in one-dimension, if the confining potential is more confining than a parabolic potential, while in two-dimension, BEC is possible for *any power law* confining potential. Using a modified semi-classical approximation, Yukalov [6] replaced the lower limit of integration from zero to a finite value given by the uncertainty relation. He thus

removed the divergence at the lower limit of the integration to obtain Bose condensation for an arbitrary power law confinement in any dimension. For a finite value of  $N$ , in contrast with the text book treatment of an ideal Bose gas, Ketterle and van Druten treated the sum directly and showed that BEC is possible even in one-dimension, the critical temperature being higher and therefore more convenient for experimental achievement of BEC in quasi-one dimensional traps [7]. In such cases, both the condensate fraction and the specific heat are smooth and continuous functions of  $T$  near  $T_c$  [7]. In two and three dimensions, the specific heat at constant particle number and for a fixed confining potential attains its maximum at the transition temperature ( $T_c$ ) and decreases rapidly thereafter. For the three dimensional case, its value approaches the classical value  $\frac{3}{2}Nk_B$  (for ideal uniform Bose gas) or  $3Nk_B$  (for non-interacting atoms trapped in a 3D harmonic oscillator well). For large  $N$ , their behavior approaches that in the thermodynamic limit [8]. Other studies also showed that BEC in lower dimensions can be obtained using a thermodynamic limit approach [9, 10].

An important physical quantity in connection with the phase transition at a critical point is the critical exponent. A critical point is characterized by divergences, discontinuities or rapid changes in thermodynamic quantities, depending on the nature of the thermodynamic quantity, dimensionality and the size of the system. The critical exponent is useful in understanding the rapidly changing behavior of thermodynamic functions at the critical temperature. In terms of a reduced temperature

$$t = \frac{T - T_c}{T_c}, \quad (3)$$

which is a dimensionless measure of the deviation of temperature from the critical temperature, one can define the critical exponent ( $\lambda$ ) for a thermodynamic function  $F(t)$  as [11]

$$\lambda = \lim_{t \rightarrow 0} \frac{\ln|F(t)|}{\ln|t|}. \quad (4)$$

This corresponds to  $F(t) \sim |t|^\lambda$ , for small values of  $|t|$ . Obviously, at the critical temperature,  $F(t)$  must either vanish or be singular, depending on the sign of  $\lambda$ . For quantities that are neither singular, nor vanish at  $T_c$ , (*e.g.*, specific heat, condensate fraction or chemical potential of a BEC containing a finite number of atoms), one can take  $F(t)$  to

be the difference of the desired thermodynamic function at  $T$  and its value at  $T_c$ . Indeed, in this case,  $\lambda$  is zero and our prescription will give  $\lambda_1$  of the relation

$$F(t) = F(0) + bt^{\lambda_1} + \dots, \quad (5)$$

for positive  $\lambda_1$  (see Eq. (2.21) of Ref. [11]). Since in this case,  $\lambda$  is zero and  $\lambda_1$  is the leading exponent of interest, the nature of the thermodynamic function near the critical point is determined by  $\lambda_1$ . Hence for such a situation, we *define*  $\lambda_1$  as *the critical exponent* ( $\lambda$ ). Therefore our modified definition is

$$\lambda = \lim_{t \rightarrow 0} \frac{\ln |F(T) - F(T_c)|}{\ln |t|}. \quad (6)$$

An important characteristic property is the universality of the critical exponent. The value of  $\lambda$  does not depend on the interatomic interaction or detailed nature of the system. Its value depends only on the dimension of the system and the symmetry of the order parameter [11]. In the present work, we calculate critical exponents for a number of thermodynamic functions of the BEC in different cases and investigate whether they depend on the system parameters. We consider a non-interacting Bose gas trapped in a harmonic oscillator potential in one, two and three dimensions. Finally, we also include realistic interatomic interactions in an approximate many-body treatment of the real Bose gas. As discussed earlier, the thermodynamic functions like specific heat, condensate fraction and chemical potential depend on the dimension of the system, the choice of the trap potential, whether interatomic interactions are included or not, *etc.* For a proper phase transition, it is expected that the critical exponents will show their universal characteristics. Our present work explores this possibility.

We consider  $N$  non-interacting bosonic atoms trapped in an isotropic harmonic potential of frequency  $\omega$ . The energy scale is so chosen that the single particle ground state is at zero energy. The energy eigenvalues  $E_n$  ( $n = 0, 1, 2, \dots$ ) are given by

$$E_n = n\hbar\omega \quad (7)$$

The average number of particles in each single particle energy eigenstate at an absolute

temperature  $T$  *i.e.*, the Bose distribution function  $f(E_n)$  is given by

$$f(E_n) = \frac{1}{e^{\beta(E_n - \mu)} - 1} \quad (8)$$

where  $\beta = 1/k_B T$  and  $\mu$  is the chemical potential. The latter is determined from the constraint that the total number of particles is  $N$ . Clearly,  $\mu$  has a temperature dependence. In general for a  $d$ -dimensional space, an energy level is degenerate with a degeneracy factor  $\gamma_n$ . The degeneracy factors for the  $n$ -th level of the one-, two- and three-dimensional harmonic oscillator are 1,  $(n+1)$  and  $\frac{(n+1)(n+2)}{2}$  respectively. The total number of bosons in the trap is fixed and can be written as

$$N = \sum_{n=0}^{+\infty} \gamma_n f(E_n) \quad (9)$$

The total energy for the system is given by

$$E(N, T) = \sum_{n=0}^{+\infty} \gamma_n f(E_n) E_n \quad (10)$$

The specific heat at fixed particle number ( $N$ ) is calculated using the relation

$$C_N(T) = \frac{\partial E(N, T)}{\partial T} \quad (11)$$

Using (8), (10), (11) one can obtain the heat capacity as

$$\begin{aligned} C_N(T) &= \beta \sum_{n=0}^{+\infty} \frac{\gamma_n E_n \exp(\beta(E_n - \mu))}{(\exp(\beta(E_n - \mu)) - 1)^2} \left[ \frac{E_n - \mu}{T} + \frac{\partial \mu}{\partial T} \right] \\ &= \beta \sum_{n=0}^{+\infty} \frac{\gamma_n E_n \exp(\beta(E_n - \mu))}{(\exp(\beta(E_n - \mu)) - 1)^2} \left[ \frac{E_n - \mu}{T} - \frac{\sum_{m=0}^{+\infty} \gamma_m (E_m - \mu) \exp(\beta(E_m - \mu)) (f(E_m))^2}{T \sum_{p=0}^{+\infty} \gamma_p \exp(\beta(E_p - \mu)) (f(E_p))^2} \right] \end{aligned} \quad (12)$$

For an ideal bosonic gas containing  $N$  bosons in a three-dimensional isotropic harmonic well, there is a well defined critical temperature  $T_c^0$  (which can be treated as a reference temperature in what follows) and is given by [3]

$$T_c^0 = \left( \frac{N}{\zeta(3)} \right)^{1/3} \frac{\hbar \omega}{k_B} \quad (13)$$

where  $\zeta(3) \simeq 1.202057$ . This is obtained from the condition that *all* the particles are in the lowest single particle state for  $T < T_c^0$ . Hence  $\mu$  remains equal to the energy of the single particle ground state for  $T < T_c^0$  and start decreasing rapidly for  $T > T_c^0$ . However as mentioned earlier, for a correct treatment [*i.e.*, not replacing the sum in

(9) by an integral], the number of bosons in the ground state becomes non-zero for *any* temperature above zero. Consequently,  $\mu$  *decreases slowly* from its maximum value (equal to the ground state energy) as  $T$  increases from zero, the rate of increase becoming suddenly rapid at some temperature close to the reference temperature  $T_c^0$  corresponding to the same value of  $N$ . Thus, in this case there is no sharp critical temperature. The heat capacity also becomes a smooth function of  $T$ , attaining a maximum at a temperature, at which  $\mu$  suddenly becomes a rapidly decreasing function of  $T$  (except for the one-dimensional case, see below). The *critical temperature*  $T_c$  is defined as the temperature at which  $C_N(T)$  is a maximum [8]

$$\left. \frac{\partial C_N(T)}{\partial T} \right|_{T_c} = 0 \quad (14)$$

For a finite value of  $N$ ,  $T_c$  is less than  $T_c^0$  and approaches  $T_c^0$  as  $N \rightarrow \infty$ . This is expected, since in this limit, the approximation of replacing the sum in (9) by an integral becomes exact. Using our adopted definition, the critical exponent for the heat capacity is given by

$$\lambda^{\text{Spht}} = \lim_{t \rightarrow 0-} \frac{\ln |C_N(t) - C_N^{\text{max}}|}{\ln |t|}, \quad (15)$$

where  $C_N^{\text{max}}$  is the maximum of  $C_N(T)$  and  $t$  is given by Eq. (3).

In a similar fashion, the critical exponents for the chemical potential and the condensate fraction is obtained. For all thermodynamic functions, we use the same critical temperature defined by (14). Since the thermodynamic functions considered here are all continuous at  $T = T_c$ , we take the difference of the value of the function at  $T$  with that at  $T_c$ . The critical exponent for the chemical potential is thus given by

$$\lambda^{\text{Chempot}} = \lim_{t \rightarrow 0+} \frac{\ln |\mu(T) - \mu(T_c)|}{\ln |t|}, \quad (16)$$

and that for the condensate fraction is

$$\lambda^{\text{Condfrac}} = \lim_{t \rightarrow 0-} \frac{\ln \left| \frac{N_0(T)}{N} - \frac{N_0(T_c)}{N} \right|}{\ln |t|}. \quad (17)$$

For the interacting case, one has to solve the  $N$ -body Schrödinger equation. An exact solution is an almost impossible task. The mean-field theory together with the assumption of a contact two-body interaction leads to the commonly used Gross-Pitaevskii equation (GPE) [12]. In this approach, all correlations are disregarded and

no realistic two-body interaction can be used. In addition, for an attractive BEC, there is a pathological singularity at the origin [13]. Instead, we adopt a simplified few-body technique, called potential harmonics expansion method (PHEM) [14, 15], in which all two-body correlations are retained (disregard of higher-order correlations is manifestly justified in a dilute BEC) and *any realistic* interatomic interaction can be incorporated. This technique has been successfully applied to both repulsive and attractive BECs [16].

In the following, we review the PHEM very briefly. Interested readers can get the details from Refs. [14, 15]. In this technique, center of mass motion is separated by introducing Jacobi coordinates, and the resulting Schrödinger equation for the relative motion is rewritten in terms of hyperspherical variables: the hyperradius ( $r$ ) and  $(3N-4)$  hyperangles, collectively denoted by  $\Omega_{N-1}$  [17]. Expansion of the relative wave function in the complete set of hyperspherical harmonics (HH) [17] leads to an *exact* treatment, but involves  $(3N-3)$  variables, which remains an almost impossible task for  $N > 3$ . We choose the subset of HH, called potential harmonics (PH) subset [18], which is sufficient for the expansion of the two-body interaction  $V(r_{ij})$  in the HH basis, where  $\vec{r}_{ij}$  is the separation vector from the  $i$ -th to the  $j$ -th particle. Since the probability of three or more particles to come close to each other is practically zero in a typical BEC (indeed this is assured in an experimental BEC, in order to avoid three-body collisions, which lead to molecule formation and consequent depletion of the BEC), correlations higher than two-body ones are practically absent. Thus the expansion of each Faddeev component of the wave function in the PH basis, which retains two-body correlations, is manifestly appropriate. Substitution of this expansion in the relative Schrödinger equation and projection on a particular PH leads to a system of coupled differential equations in  $r$ . The latter is solved by hyperspherical adiabatic approximation [19], to obtain a large number of energy eigenvalues of the entire BEC for different orbital angular momenta of the system. These are then used in Eq.(9) to calculate the chemical potential at a chosen temperature  $T$ . Note that in this case, the sums over  $n$  in Eqs.(9), (10) and (12) are to be replaced by double sums over  $\{n, l\}$  and  $\gamma_n$  is replaced by  $(2l+1)$ . The energy eigenvalue  $E_{nl}$  is the energy of the  $n$ -th excitation for the orbital angular momentum  $l$  of the system. Finally Eq.(12) is used to calculate  $C_N(T)$  [20]. Condensate fraction is obtained as  $\frac{N_0}{N}$ , where  $N_0$  is the  $n=0$  term of Eq.(9).

Initially, we consider non-interacting bosons in an isotropic harmonic trap in one, two and three dimensions. We first present the results for  $N$  non-interacting trapped bosons in a one-dimensional harmonic oscillator potential. In panel (a) of Fig. 1, we plot

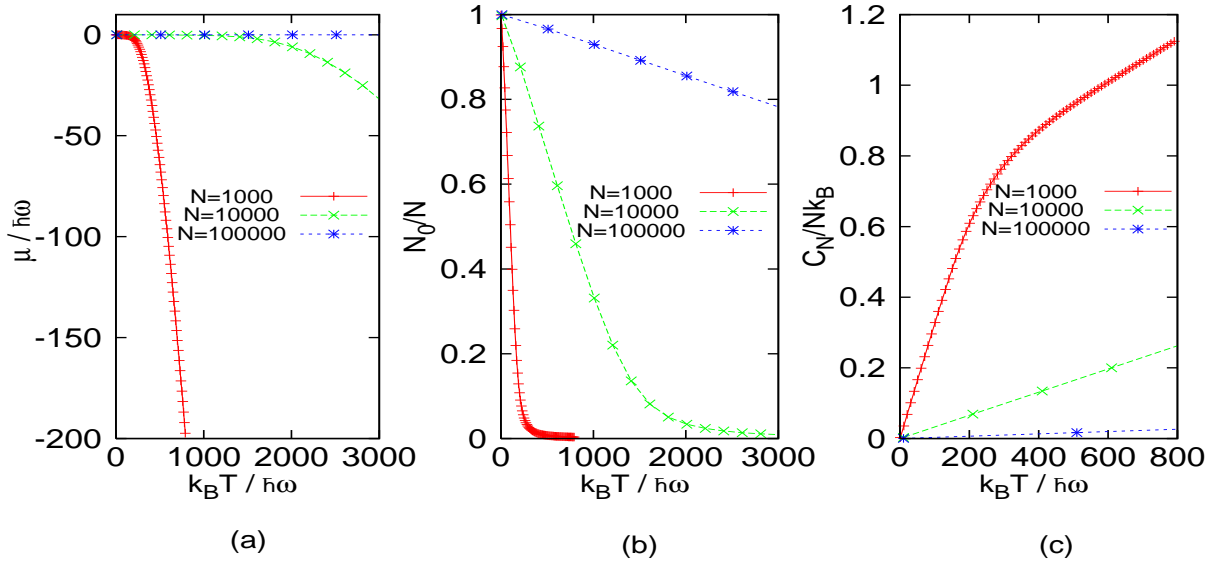


FIG. 1. (Color online) Plot of chemical potential (panel a), condensate fraction (panel b) and  $C_N / N k_B$  (panel c) as functions of  $k_B T / \hbar \omega$  for indicated number ( $N$ ) of non-interacting bosons in a one-dimensional harmonic trap. Note that chemical potential and  $k_B T$  are expressed in energy oscillator unit ( $\hbar \omega$ ). Note also that  $C_N / N k_B$  is a *monotonically increasing* function of  $T$ , giving rise to the common notion that there is no criticality in one dimension.

$\mu / \hbar \omega$  against  $k_B T / \hbar \omega$  for  $N = 1000, 10000$  and  $100000$ . For  $N = 1000$ , one notices that  $\mu$  remains nearly equal to its value at  $T = 0$  up to  $k_B T / \hbar \omega \sim 200$ , after which  $\mu$  starts decreasing rapidly. This indicates a critical behavior. For larger number of particles, such a rapid change occurs at a much higher temperature, indicating that the critical temperature increases rapidly with  $N$ . This observation agrees with Ref. [7]. In panel (b) of Fig. 1, we plot the condensate fraction, as a function of  $k_B T / \hbar \omega$  for the chosen values of  $N$ . Once again, we notice that  $\frac{N_0}{N}$  has a rapid change at  $k_B T / \hbar \omega \sim 200$  for  $N = 1000$ , while for larger  $N$  the rapid change occurs at a much higher temperature. All



these indicate that there is a continuous phase transition at the critical point. However, from a plot of  $C_N/Nk_B$  against  $k_B T/\hbar\omega$  in panel (c) of Fig. 1, we notice that for the one-dimensional case,  $C_N(T)$  is a monotonic function of  $T$  and so definition (14) cannot be used for calculation of  $T_c$ . This implies that the one-dimensional case is distinctly different from higher dimensions (described later). This is reflected in the text book treatment (where sums are replaced by integrals), for which  $T_c^0$  diverges since  $\zeta(1)$  diverges (compare with Eq. (13) and Ref. [3]). Instead, we also notice from Fig. 1(c), that  $C_N(T)$  is almost linear for  $T$  far away from the critical region with distinctly different slopes. Hence for the one-dimensional case only, we define  $T_c$  as the temperature at which the extrapolated asymptotic straight lines intercept. This gives  $T_c = 243.77$  nK for  $N = 1000$ . For  $N = 10000$  and  $100000$   $T_c$  is larger (2132.5 nK and 15888 nK 100000 respectively). The slopes of the  $C_N(T)/Nk_B$  versus  $k_B T/\hbar\omega$  curve before and after the critical point become closer to each other, as the particle number increases. Consequently, criticality gets smeared out as  $N$  increases. Although existence of a continuous phase transition for *finite*  $N$  in a one-dimensional harmonic oscillator trap is in sharp contrast with common acceptance, the fact that the criticality gradually fades away as  $N \rightarrow \infty$  is in agreement. Using this value of  $T_c$  for a particular value of  $N$ , we plot  $\ln |\mu(T) - \mu(T_c)|$ ,  $\ln |\frac{N_0(T)}{N} - \frac{N_0(T_c)}{N}|$  and  $\ln |(C_N(T) - C_N(T_c))/Nk_B|$  against  $\ln |t|$  in panels (a), (b) and (c) of Fig. 2 respectively, for the one dimensional trap. As expected, we obtain straight line fits. All such straight line fits for different  $N$  are parallel (to within calculation errors), showing that the critical exponents are independent of  $N$ . A typical straight line fit for each plot is shown by a black dotted line. The slopes of these lines for panels (a), (b) and (c) are respectively 0.919, 1.122 and 1.004. These are then the critical exponents for chemical potential, condensate fraction and specific heat in the one dimensional harmonic oscillator trap.

We next repeat the calculations for a two-dimensional isotropic harmonic oscillator trap. In this case,  $C_N(T)$  versus  $T$  curve has a maximum, from which we calculate  $T_c$  according to Eq. (14). Panels (a), (b), (c) of Fig. 3 present plots of  $\ln |(\mu(T) - \mu(T_c))/\hbar\omega|$ ,  $\ln |N_0(T)/N - N_0(T_c)/N|$  and  $\ln |(C_N(T) - C_N(T_c))/Nk_B|$  respectively as functions of  $\ln |t|$  for  $N = 1000, 10000, 100000$  and  $1000000$  atoms in the

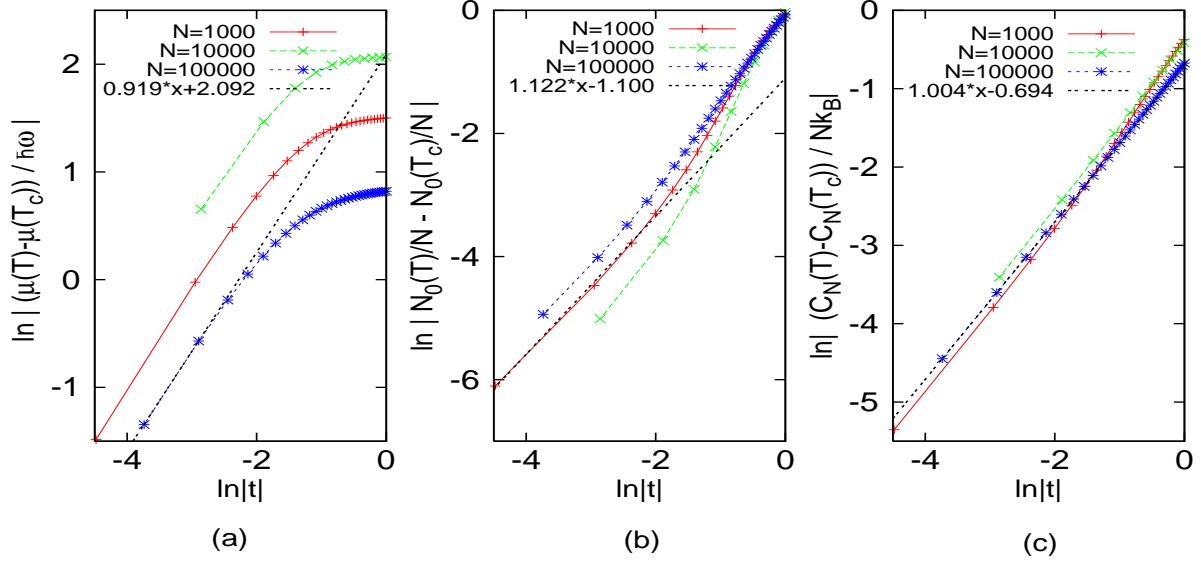


FIG. 2. (Color online) Plot of  $\ln |(\mu(T) - \mu(T_c))/\hbar\omega|$  (panel a),  $\ln |N_0(T)/N - N_0(T_c)/N|$  (panel b) and  $\ln |(C_N(T) - C_N(T_c))/Nk_B|$  (panel c) as functions of  $\ln |t|$  for indicated number ( $N$ ) of non-interacting bosons in a one-dimensional harmonic trap. Typical straight line fits to the asymptotic regions are shown by black dotted lines. Parameter values of the fits are also indicated. Note that the asymptotic linear regions for different  $N$  are parallel, showing that the critical exponent of a particular thermodynamic function is independent of  $N$ .

trap. The asymptotic part of the plots are straight lines, which are parallel for different  $N$  of the same panel. This again indicates that the critical exponents are independent of  $N$  (note that a few points for large negative  $\ln |t|$  in panel (c) do not fall on the straight lines due to numerical errors, and have been disregarded). Typical straight line fits are indicated by black dotted lines and the fitting parameters are also displayed. From these, we find that the critical exponents for chemical potential, condensate fraction and specific heat are 0.915, 1.006 and 1.810 respectively for the two dimensional trap.

We next calculate these quantities for a three dimensional harmonic oscillator trap. In Fig. 4, we plot  $\mu/\hbar\omega$ , condensate fraction and  $C_N/Nk_B$  as functions of  $T/T_c^0$  in panels (a), (b) and (c) respectively, where  $T_c^0$  is calculated from Eq. (13). Since in

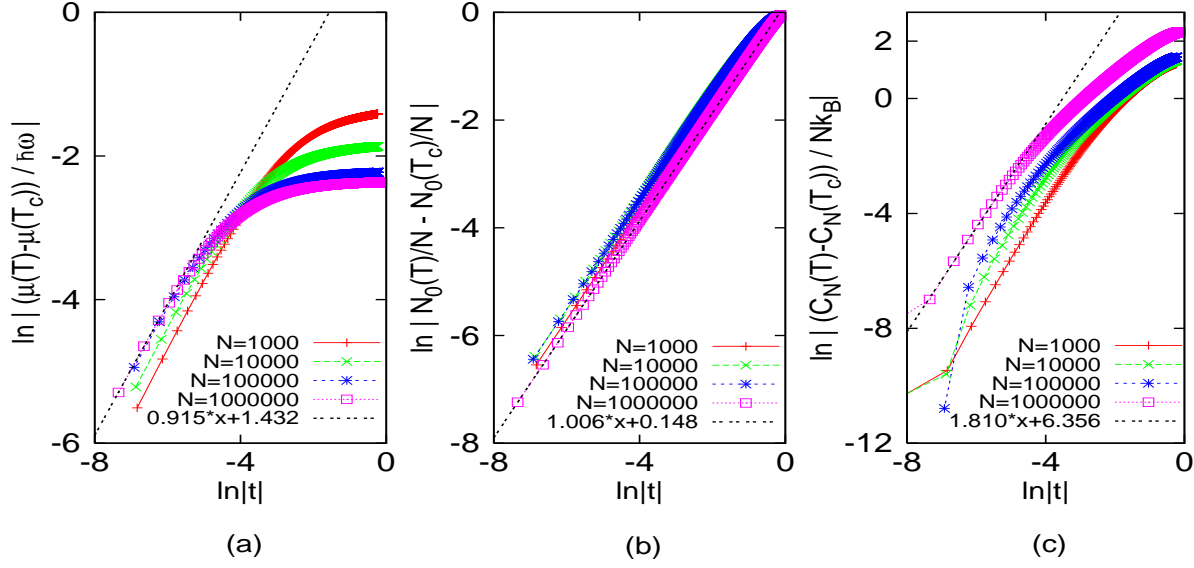


FIG. 3. (Color online) Plot of  $\ln |(\mu(T) - \mu(T_c))/\hbar\omega|$  (panel a),  $\ln |N_0(T)/N - N_0(T_c)/N|$  (panel b) and  $\ln |(C_N(T) - C_N(T_c))/Nk_B|$  (panel c) as functions of  $\ln |t|$  for indicated number ( $N$ ) of non-interacting bosons in a two-dimensional harmonic trap. For other comments, see caption of Fig. 2.

this case,  $C_N(T)$  shows a clear maximum, we can easily calculate corresponding  $T_c$  from Eq. (14). Calculated values of  $T_c/T_c^0$  are 0.896, 0.949, 0.976, and 0.989 for  $N = 1000$ , 10000, 100000 and 1000000 respectively. Using these values of  $T_c$ , we calculate the differences of the three thermodynamic functions from their values at the critical points and plot the logarithm of their moduli against  $\ln |t|$  in panels (a), (b) and (c) of Fig. 5. All calculated points for the chemical potential fall nicely on straight lines in the interval  $-10 < \ln |t| < -6$ . Interestingly, all curves for different  $N$  fall on the *same straight line* in this  $\ln |t|$  interval. This shows that the corresponding critical exponent is independent of  $N$ . For  $\ln |t|$  increasing above  $-6$ , the curves for different  $N$  separate gradually, showing that the nature of  $\mu$  versus  $T$  curves *away from the critical point* depend on  $N$ . On the other hand, numerical errors start to show up for  $\ln |t| < -10$ . Hence such points are ignored for calculation of the critical exponent. The same behavior is seen for condensate fraction [Fig. 5(b)] and specific heat [Fig. 5(c)]. However the numerical errors are larger

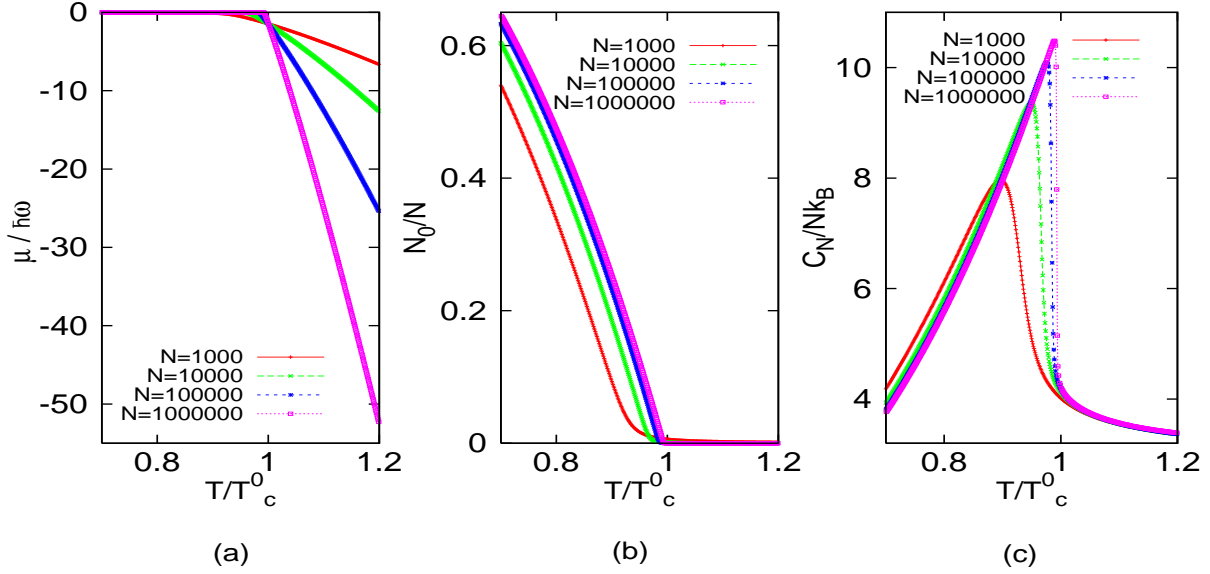


FIG. 4. (Color online) Plot of chemical potential (panel a), condensate fraction (panel b) and  $C_N/Nk_B$  (panel c) as functions of  $T/T_c^0$  [where  $T_c^0$  is defined in Eq. (13)] for indicated number ( $N$ ) of non-interacting bosons in a three-dimensional isotropic harmonic trap. Note that the chemical potential is expressed in energy oscillator unit ( $\hbar\omega$ ). In contrast with Fig. 1(c), panel (c) above shows a maximum followed by a sharp drop in  $C_N$ , the sharpness of the drop increasing with  $N$ . Furthermore, as  $N \rightarrow \infty$ , the temperature at which this sharp drop occurs approaches  $T_c^0$ .

for the calculation of the specific heat, showing some divergences. In the last case, the linear parts for different  $N$  are nearly parallel showing that the corresponding critical exponent is independent of  $N$ . Typical straight line fits are shown by black dotted lines in Fig. 5, together with the fitted parameter values. Calculated critical exponents for chemical potential, condensate fraction and specific heat are 0.913, 0.995 and 1.900 respectively. These values are independent of  $N$ , confirming universality of the critical exponent.

Next we present the results of our calculation for  $N$  bosons interacting via the

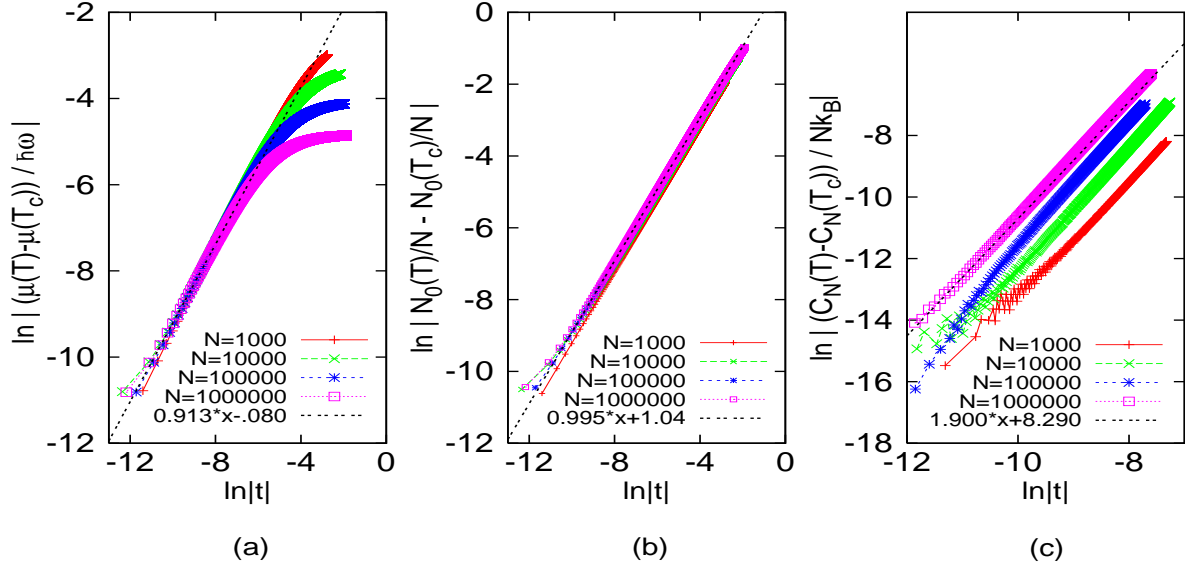


FIG. 5. (Color online) Plot of  $\ln |(\mu(T) - \mu(T_c))/\hbar\omega|$  (panel a),  $\ln |N_0(T)/N - N_0(T_c)/N|$  (panel b) and  $\ln |(C_N(T) - C_N(T_c))/Nk_B|$  (panel c) as functions of  $\ln |t|$  for indicated number ( $N$ ) of non-interacting bosons in a three-dimensional harmonic trap. For other comments, see caption of Fig. 2.

van der Waals potential and trapped in an isotropic three dimensional harmonic oscillator potential. As mentioned earlier, the GPE uses a contact interaction, whose strength depends *only* on the  $s$ -wave scattering length  $a_s$  and hence is shape independent of the two-body potential. An earlier calculation [21] showed that calculated observables are indeed *shape dependent*. Hence it is appropriate that a finite-range realistic interatomic potential like the vdW potential should be used. Calculated values of  $\ln |\mu(T) - \mu(T_c)|$ ,  $\ln |\frac{N_0(T)}{N} - \frac{N_0(T_c)}{N}|$  and  $\ln |C_N(T) - C_N(T_c)|$  [20] have been plotted against  $\ln |t|$  in panels (a), (b) and (c) of Fig. 6. Once again these plots are straight lines in the asymptotic ( $\ln |t| \rightarrow -\infty$ ) region. However, due to large error in the numerical calculation, especially when  $|t|$  is very small there are some fluctuations for large negative  $\ln |t|$ . Hence the lower limit of  $\ln |t|$  has been restricted to  $-8$ . From the linear plots we get the values of  $\lambda^{\text{Chempot}} = 0.913$ ,  $\lambda^{\text{Condfrac}} = 1.005$  and  $\lambda^{\text{Sph}} = 1.878$ . We notice that these values are independent of  $N$  (since the asymptotic linear parts are parallel for different  $N$ ,

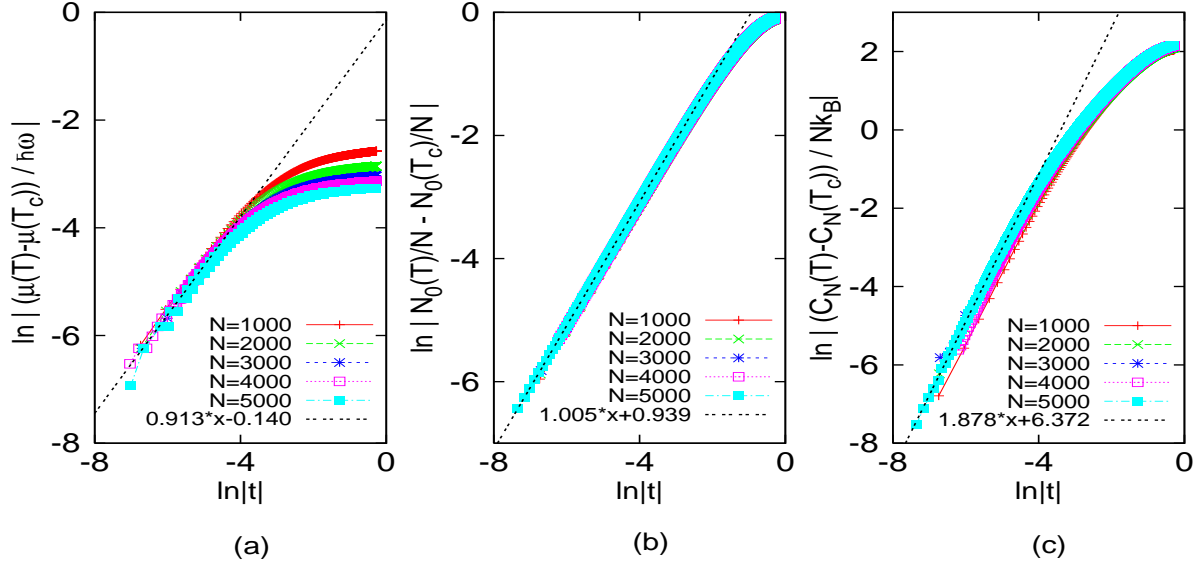


FIG. 6. (Color online) Plot of  $\ln |(\mu(T) - \mu(T_c))/\hbar\omega|$  (panel a),  $\ln |N_0(T)/N - N_0(T_c)/N|$  (panel b) and  $\ln |(C_N(T) - C_N(T_c))/Nk_B|$  (panel c) as functions of  $\ln |t|$  for indicated number of atoms interacting via van der Waals potential in a three-dimensional harmonic trap. For other comments, see caption of Fig. 2.

in each of the panels (a), (b) and (c) of Fig. 6. We also notice that the values of the critical exponents for the interacting three dimensional case are equal (within numerical errors) to the corresponding critical exponents for the non-interacting three dimensional case. These observations show that the critical exponents of the BEC obey universality property.

In conclusion, we remark that a continuous phase transition occurs in a non-interacting Bose gas trapped in a harmonic oscillator potential *even in one-dimension*. This is in sharp contrast with standard text book results [1, 3], but in agreement with Ketterle and van Druten [7]. This is because the replacement of sums over single particle states by an integral over energy is not a valid approximation for *discrete* energy levels, especially when  $N$  is small. However, we find that there are some distinct characteristic features in the one-dimensional case. For example, the specific heat is a monotonic

increasing function, whereas in higher dimensions it has a maximum. The dependence of chemical potential, condensate fraction and specific heat on temperature for the non-interacting one-dimensionally trapped Bose gas has similar behavior as when a Bose gas interacts through the harmonic Calogero interaction [22]. We prescribe a way to calculate the transition temperature ( $T_c$ ) in this case (as the temperature at which the asymptotically linear curves for  $T$  far below and far above the critical region intercept). As  $N$  increases,  $T_c$  increases *rapidly*. For  $N \rightarrow \infty$ , the two asymptotic straight lines have the same slope and no  $T_c$  can be defined. This agrees with the commonly accepted idea that there is no Bose-Einstein condensation in one-dimension. Ketterle and van Druten argued that a quasi one-dimensional experimental setup is advantageous, since  $T_c$  becomes larger [7]. However, in view of the above discussion, it is clear that such an experimental setup is useful for relatively small  $N$ . Although  $T_c$  increases rapidly with  $N$ , the difference in the slopes of the two asymptotic regions of  $C_N(T)$  versus  $T$  curve decreases. Hence the difference in the properties of the two phases gradually reduce, making a distinct phase transition obscure.

We have also calculated the critical exponents for three thermodynamic functions, *viz.* chemical potential, condensate fraction and specific heat at constant particle number for a non-interacting Bose gas in an isotropic harmonic trap in one, two and three dimensions. For the three dimensional case, we have also included a realistic two-body interaction. We find that the critical exponent for a particular thermodynamic function does not depend on  $N$  or whether the bosons are interacting or not. It depends on the type of the thermodynamic function and the dimensionality of the space. This is in agreement with the idea of universality.

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